Article

Independent Domination in Claw-Free Cubic Graphs

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Abstract: A vertex set S of a graph G is called an independent dominating set if S is an independent set and each vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The independent domination number $i(G)$ of G is the minimum cardinality of an independent dominating set in G. This paper first proves that if G is a connected $K_{1,3}$-free cubic graph, then $i(G) \leq \frac{1}{3}|V(G)|$. Meanwhile, $i(G) = \frac{1}{3}|V(G)|$ if and only if $G \in \mathcal{H}$, where $\mathcal{H}$ is an infinite cubic family with each graph being a C_6^+-necklace. Then, it is shown that if G is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph with no $C_3 \Box K_2$-component, then $i(G) \leq \frac{5}{18}|V(G)|$. This result is tight.

Keywords: independent domination; claw-free; cubic graphs

MSC: 05C69; 05C07

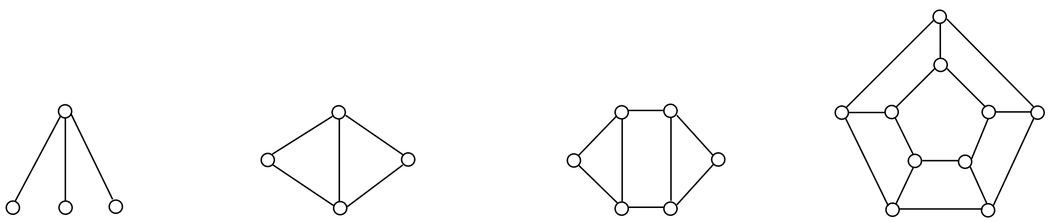
1. Introduction

Let $G = (V, E)$ be a graph. A set S of vertices in a graph G is called an *independent dominating set* , abbreviated as ID-set, if S is an independent set and every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The *independent domination number* of G, denoted as $i(G)$, is the minimum cardinality of an ID-set, and an ID-set of cardinality $i(G)$ in G is called an $i(G)$-set. For recent books on independent domination, please refer to [1, 2].

The notations and graph theory terminologies in this paper generally follow [3]. The *degree* of a vertex in graph G is denoted as $d_G(v)$, abbreviated as $d(v)$. For an integer $k \geq 1$, a k^+-vertex is a vertex having a degree of at least k. The maximum degree among the vertices of G is denoted as $\Delta(G)$. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, the subgraph induced by S is denoted as $G[S]$, and G-S is abbreviate as $G[V(G) \setminus S]$. Let $[k]$ be the set $\{1, 2, \ldots, k\}$ for a positive integer k. A cycle on n vertices is denoted as C_n. For vertex $v \in V(G)$, let u be a neighbor of v and u be on a triangle T. Then, v is *adjacent* to triangle T, and u is *incident* with T.

A graph is *F-free* if it does not include F as an induced subgraph. A *claw* is a star $K_{1,3}$. A *diamond* is a $K_4 - e$, where e is referred to as the missing edge. In this paper, $K_4 - e$ is abbreviated as K_4^-. C_6^+ is defined for the simple graph obtained from two vertex disjoint triangles by adding two vertex disjoint edges to it. In Figure 1, from left to the right, the first three subgraphs are $K_{1,3}$, K_4^-, and C_6^+, respectively. A *k-regular* graph is a graph where every vertex has a degree of k. If $k = 3$, then the graph is a *cubic* graph. Independent domination in cubic graphs and claw-free graphs has been extensively investigated in the literature (e.g., [3–8], etc.).

For a connected k-regular graph G where $k \geq 1$. Rosenfeld [9] pointed out that $i(G) \leq \frac{V(G)}{2}$, and this is tight only for the balanced complete bipartite graph $K_{k,k}$. For a cubic graph, Lam, Shiu, and Sun [10] established the following upper bound on the independent domination number.



(a) (b) (c) (d)

Figure 1. Three subgraphs and $C_5 \Box K_2$.

Theorem 1.1. *[10] If G is a connected cubic graph of order n, where $n \geq 8$, then $i(G) \leq 2n/5$, and the bound is tight by $C_5 \square K_2$. See Figure 1d.*

Goddard and Henning [3] speculated that there is only one graph where the upper bound in Theorem 1.1 is tight. Dorbec et al. [11] claimed that the conjecture holds if, in addition, G does not have a subgraph isomorphic to $K_{2,3}$.

Theorem 1.2. *[11] If $G \neq C_5 \square K_2$ is a connected cubic graph of order n that does not have a subgraph isomorphic to $K_{2,3}$, then $i(G) \leq 3n/8$.*

The number of vertices in the largest independent set of a graph G is referred to as the independent number and is denoted as $\beta(G)$. Murugesan et al. [12] pointed out that an independent set of a graph G is dominating if and only if it is maximal. Thus, $\beta(G)$ is a natural upper bound of G on the independent dominating number. Wang [13] obtianed the exact values for $\{K_{1,3}, K_4^-\}$-free cubic graphs on the independent number.

Theorem 1.3. *[13] For every connected claw-free cubic graph G of order n, if G ($G \neq K_4$) includes no K_4^- as an induced subgraph, then $\beta(G) = n/3$.*

Therefore, if G is a $\{K_{1,3}, K_4^-\}$-free cubic graph, then $i(G) \leq \frac{|V(G)|}{3}$. In this paper, it is shown that if G is a connected $K_{1,3}$-free cubic graph, then $i(G) \leq \frac{1}{3}|V(G)|$, and this bound is tight. Furthermore, it is proven that if G is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph with no $(C_3 \square K_2)$-component, then $i(G) \leq \frac{5}{18}|V(G)|$, and this bound is best possible.

2. Main Results

A graph G is said to be *subcubic* if its maximum degree is three. Let $n_j(G)$ be the number of vertices of degree j in G. For a graph H, an H*-unit* in G is called an induced subgraph of G that is isomorphic to H. An edge e is called a *triangle edge* if it is on a triangle, and otherwise, e is a *flat edge*.

Definition 2.1. A graph G is SP-1 if $G = K_3$ or the following three properties hold: (i) G is connected, (ii) $\Delta(G) = 3$, and (iii) every vertex belongs to a triangle.

Theorem 2.1. *If G is an SP-1 graph, then $3i(G) \leq n_2(G) + n_3(G)$.*

An infinite family $\mathcal{H}$ with an independent domination number $\frac{1}{3}$ of its order can be established as follows. Given k disjoint copies $F_1, F_2, \ldots, F_k$ of C_6^+, where $V(F_i) = \{o_i, f_i, g_i, h_i, p_i, q_i\}$, and $o_i f_i g_i h_i p_i q_i o_i$ is a 6-cycle with two chords $f_i q_i$ and $g_i p_i$. Let H_k be acquired from the disjoint union of these k C_6^+ by adding the edges $\{h_i o_{i+1} : i \in [k-1]\}$ and $h_k o_1$. When $k = 1$, $H_1 = C_3 \square K_2$. Let $\mathcal{H} = \{H_k : k \geq 1 \textit{ be an integer}\}$.

Theorem 2.2. *If G is a connected claw-free cubic graph, then $i(G) \leq \frac{1}{3}|V(G)|$. Meanwhile, $i(G) = \frac{1}{3}|V(G)|$ if and only if $G \in \mathcal{H}$.*

For every C_6^+-unit F_i in H_k, $i \in [k]$, $V(F_i)$ contains at least two vertices in an ID-set of H_k. Therefore, $i(H_k) \geq \frac{1}{3}|V(H_k)|$. Let $S = \bigcup_{i=1}^{k}\{f_i, p_i\}$, then S is an ID-set in H_k. So, $i(H_k) = \frac{1}{3}|V(H_k)|$. For instance, when $k = 4$, a minimum ID-set of H_4 is illustrated in Figure 2.

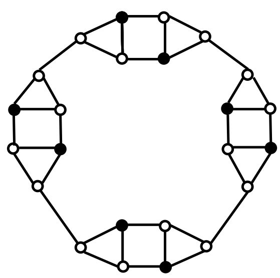


Figure 2. An $i(H_4)$-set of H_4 indicated with darkened vertices.

Definition 2.2. Let G be a graph, with $G_1, G_2, \ldots, G_k$ being the components of G. A graph G is SP-2 if the following three properties hold for each $i \in [k]$: (i) $\Delta(G_i) = 3$, (ii) every vertex belongs to a triangle in G_i, and (iii) G_i has no K_4^--unit or C_6^+-unit.

Theorem 2.3. *If G is an SP-2 graph without $(C_3 \square K_2)$-component, then $18i(G) \leq 8n_2(G) + 5n_3(G)$.*

Corollary 2.4. *If G is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph without $(C_3 \square K_2)$-component, then $i(G) \leq \frac{5}{18}|V(G)|$.*

This bound is tight, and Figure 3 shows the darkened vertices form an $i(G)$-set of G of cardinality $\frac{5}{18}|V(G)|$.

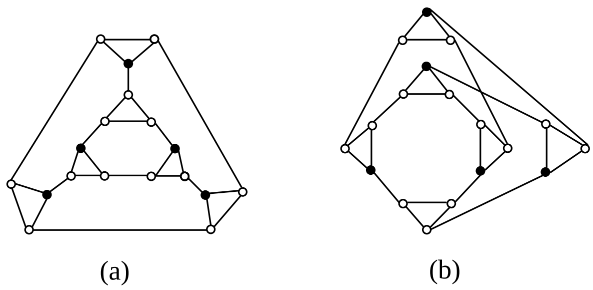


Figure 3. Two cubic graphs with independent domination number $\frac{5}{18}$ of their orders.

3. Proof of Theorem 2.1 and Theorem 2.2

First, we will want to prove Theorem 2.1, which is stated as follows: If G is an SP-1 graph, then $3i(G) \leq n_2(G) + n_3(G)$.

Proof. By means of contradiction, let G be a counterexample of an SP-1 graph with a minimum order. Then, G is connected, and $3i(G) > n_2(G) + n_3(G)$. Let $G = (V(G), E(G))$ and $|V(G)| = n$. We proceed further with several claims.

Claim 1. $n \geq 7$.

Proof. If $n = 3$, $G = K_3$ is not a counterexample. If $n = 4$, G is either K_4 or K_4^-. In either of cases, we have $3i(G) = 3 \leq n_2(G) + n_3(G)$.

If $n = 5$, let v be the vertex of degree three, v_1, v_2, let v_3 be the three neighbors of v in G, and let w be the vertex not adjacent to v. Since G is an SP-1 graph, w belongs to a triangle, i.e., at least two vertices in $\{v_1, v_2, v_3\}$ with w will form triangles. Without loss of generality, it could be assumed that v_1, v_2, and w form a triangle. Then, $G[\{v, v_1, v_2, w\}]$ is diamond unit since $\Delta(G) = 3$, v_1v_3, $v_2v_3 \notin E(G)$. That is, the triangle incident v_3 does not contain v_1 or v_2. Then, the unique possibility is that vwv_3 forms a triangle. However, it contradicts that $vw \notin E(G)$, i.e., v_3 does not belong to any triangle, showing a contradiction. Hence, there is no SP-1 graph when $n = 5$.

If $n = 6$, then G is one of the three graphs shown in Figure 4. In each of the cases, $3i(G) = 6 \leq n_2(G) + n_3(G)$. Therefore, $n \geq 7$. □

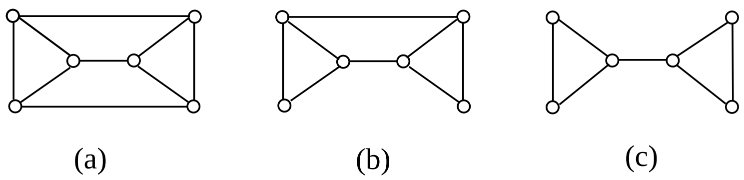


Figure 4. The three special subcubic graphs of order 6.

Claim 2. Let $X \subseteq V(G)$, $G' = G - X$. If every component of G' is an SP-1 graph with an order of less n, then $3i(G') \leq n_2(G') + n_3(G')$.

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of G', and let D_i be an $i(G_i)$-set, $i \in [k]$. By linearity, we have

$$3i(G') = 3\sum_{i=1}^{k} i(G_i) \leq \sum_{i=1}^{k}(n_2(G_i) + n_3(G_i)) = n_2(G') + n_3(G').$$

□

Claim 3. G contains no diamond unit.

Proof. Suppose that G has a diamond unit S, where $V(S) = \{a, b, c, d\}$ and ab are the missing edges. Since G is connected and $n \geq 7$, at least one of a, b has a degree of three.

If a or b has a degree of two, without loss of generality, say $d(b) = 2$. Then, $d(a) = 3$, and a is adjacent to a triangle. Let $G' = G - V(S)$, and let D' be an $i(G')$-set. Note that G' is still an SP-1 graph, and $D' \cup \{c\}$ is an ID-set of G. Therefore, $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 4 < 3i(G)$, showing a contradiction.

So, $d(a) = d(b) = 3$, and both of a and b are adjacent to triangles. Let $G' = G - V(S)$, and D' be an $i(G')$-set. Then, $D' \cup \{c\}$ is an ID-set of G. Note that G' contains at most two components, each of which is SP-1. By Claim 2, $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 4 < 3i(G)$, showing a contradiction. □

Claim 4. G contains no C_6^+-unit.

Proof. Suppose that G has a C_6^+-unit F, where $V(F) = \{o, f, g, h, p, q\}$, and $ofghpqeo$ has two chords fq and gp. At least one of o and h has a degree of three since $n \geq 7$.

If o or f has a degree of two, say f, then $d(o) = 3$, and o is adjacent to a triangle. Let $G' = G - V(F)$ and let D' be an $i(G')$-set. Noting that G' is SP-1, and $D' \cup \{f, p\}$ is an ID-set of G. Then, $3i(G) \leq 3(i(G') + 2) \leq 6 + n_2(G') + n_3(G') = 6 + n_2(G) + n_3(G) - 6 < 3i(G)$, showing a contradiction. Thus, $d(o) = d(h) = 3$. Let $G' = G - V(F)$ and D' be an $i(G')$-set. Then, G' contains at most two components, each of which is SP-1, and $D' \cup \{f, p\}$ is an ID-set of G. By Claim 2, $3i(G) \leq 3(i(G')+2) \leq 6+n_2(G')+n_3(G') = 6+n_2(G)+n_3(G)-6 < 3i(G)$, showing a contradiction. □

Claim 5. G is cubic.

Proof. Let T be a triangle containing at least one vertex of degree two. Since $n \geq 7$, T has at most two vertices of degree two.

If T has two vertices, say y and z, such that $d(y) = d(z) = 2$, then the graph $G' = G - V(T)$. Let D' be an $i(G')$-set. Note that G' is still an SP-1 graph, and $D' \cup \{y\}$ is an ID-set of G. Then, $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 3 < 3i(G)$, showing a contradiction.

Next, consider the case that T has a unique vertex of degree two. Let $V(T) = \{x, y, z\}$ and $d(z) = 2$, $d(x) = d(y) = 3$. By Claim 3, x and y do not have an incident triangle other than T. Thus, each vertex in $G' = G - V(T)$ is still contained in a triangle. This indicates that each component of G' is also SP-1. Let D' be an $i(G')$-set, then $D' \cup \{z_1\}$ is an ID-set of G. Therefore, $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 3 < 3i(G)$, showing a contradiction. □

According to Claims 3–5 and the definition of SP-1 graph, G is a $\{K_{1,3}, diamond, C_6^+\}$-free cubic. Let T_1 be a triangle in G, where $V(T_1) = \{x_1, y_1, z_1\}$. Suppose that x_2, x_3, and x_4 are not on T_1 and are the neighbors of x_1, y_1, and z_1, respectively. By Claim 3, x_2, x_3, and x_4 are distinct. Also, by Claim 4, any two of x_2, x_3, and x_4 do not contain a triangle. Therefore, x_1, y_1 and z_1 are adjacent to three different triangles, say T_2, T_3, and T_4, respectively. Let $V(T_i) = \{x_i, y_i, z_i\}$, $i = 2, 3, 4$, and $\{x_1x_2, y_1x_3, z_1x_4\} \subseteq E(G)$. Let $G' = G - V(T_1)$, and let D' be an $i(G')$-set. Note that every component of G' is SP-1. If $\{x_2, x_3, x_4\} \subseteq D'$, then D' is also an ID-set of G. Thus, $3i(G) \leq 3i(G') \leq n_2(G') + n_3(G') = n_2(G) + n_3(G) - 3 < 3i(G)$, showing a contradiction. So, without loss of generality, it could be assumed $x_2 \notin D'$. Then, $D' \cup \{x_1\}$ is an ID-set of G, and $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 3 < 3i(G)$, showing a contradiction. □

Now we give the proof of Theorem 2.2.

Proof. Let $n = |V(G)|$, since G is a claw-free cubic graph, then G is an SP-1 graph, and each vertex of $V(G)$ has a degree of 3. Thus, by Theorem 2.1, $i(G) \leq \frac{1}{3}|V(G)|$.

Next, the extreme graphs are described, i.e., $i(G) = \frac{n}{3}$. Since G is a cubic graph, n is even. When $n = 4$, $G = K_4$, and $i(G) = \frac{n}{4}$, there is a contradiction. When $n = 6$, G is either $C_3 \square K_2$ or $K_{3,3}$, and since G is claw-free, we have $G = C_3 \square K_2$; in this case, $G \in \mathcal{H}$. Thus, $n \geq 8$.

Claim 1. G has no diamond unit.

Proof. Suppose there is a diamond unit S, where $V(S) = \{a, b, c, d\}$ and ab is the missing edge in S. Let x be the neighbor of a not in S, and let y be the neighbor of b not in S. Then, $x \neq y$, and otherwise, x does not belong to a triangle unit.

If x and y are not adjacent, then x and y belong to different triangles. Let G' be obtained from $G - V(S)$ by adding the flat edge xy. Thus, G' is an SP-1 cubic graph. Let D' be an $i(G')$-set of G', so $|D'| = i(G') \leq \frac{n-4}{3}$. At most one of x and y is contained in D' since $xy \in E(G)$. If $x \in D'$ and $y \notin D'$, let $D = D \cup \{b\}$. If $y \in D'$ and $x \notin D'$, let $D = D' \cup \{a\}$. If $x, y \notin D'$, let $D = D' \cup \{c\}$. In each of these cases, the set D is an ID-set of G, and $|D| = |D'| + 1$, indicating that $i(G) \leq |D'| + 1 \leq \frac{n-1}{3} < \frac{n}{3}$, a contradiction.

If x and y are adjacent in G, then they belong to a common triangle unit T in G. Let $G' = G - V(S)$, and G' is an SP-1 graph. Let D' be an $i(G')$-set of G', $|D'| = i(G') \leq \frac{n-4}{3}$. Note that $D' \cup \{c\}$ is an ID-set of G, so $i(G) \leq |D| = |D| + 1 \leq \frac{n-4}{3} + 1 = \frac{n-1}{3} < \frac{n}{3}$, showing a contradiction. Thus, G has no diamond unit. □

Claim 2. Every triangle unit in G belongs to a C_6^+-unit.

Proof. Suppose that a triangle unit T_1 is not in any C_6^+-unit, where $V(T_1) = \{x_1, y_1, z_1\}$. Since G is diamond-free and $n \geq 8$, then any two vertices in $V(T_1)$ have no common neighbor except the vertex in $V(T_1)$. Meanwhile, since T_1 is not contained in any C_6^+-unit, then any two vertices in $V(T_1)$ are not adjacent to a common triangle except for T_1. Thus, it is assumed that x_1 is adjacent to a triangle T_2, y_1 is adjacent to T_3, and z_1 is adjacent to T_4, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{2, 3, 4\}$, and $\{x_1x_2, y_1x_3, z_1x_4\} \subseteq E(G)$. Let $W = \{y_2, z_2, y_3, z_3, y_4, z_4\}$. This proof will be completed by considering the following three cases.

Case 1. $e(G[W]) \geq 5$. If $e(G[W]) = 6$, then G is determined, and $i(G) = 3 = \frac{n}{4}$, contradicting that $i(G) = \frac{n}{3}$. Therefore, $e(G[W]) = 5$. Let e_1 and e_2 be two edges in $G[W]$ and not be y_2z_2, y_3z_3, or y_4z_4. There are two distributions of e_1 and e_2, as demonstrated in Figure 5. Let $G' = G - \cup_{i=1}^{4} V(T_i)$, and let D' be an $i(G')$-set, so every components of G' is SP-1. We have $i(G') \leq \frac{1}{3}(n - 12) = \frac{n}{3} - 4$ by Theorem 2.1. Note that $D' \cup \{x_2, x_3, x_4\}$ is an ID-set of G. Thus, $i(G) \leq D' + 3 = \frac{n}{3} - 1 < \frac{n}{3}$, showing a contradiction.

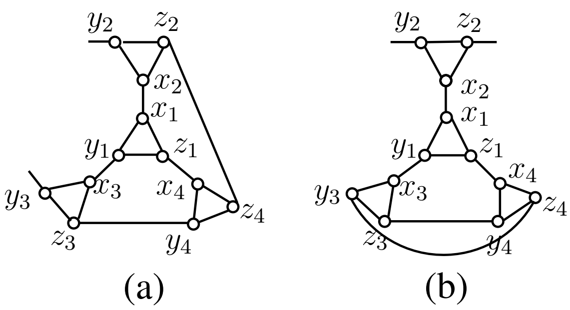


Figure 5. The graphs for Case 1 when $e(G[W]) = 5$.

Case 2. $e(G[W]) = 4$. By symmetry, it is assumed that $z_3y_4 \in E(G)$. Let x be the neighbor of y_2 not in W, let y be the neighbor of z_2 not in W, let z be the neighbor of y_3 not in W, and let r be the neighbor of z_4 not in W. Since $d(x) = 3$, at least one vertex in $\{y, z, r\}$ is not adjacent to x, and it is assumed that $xr \notin E(G)$. Let G' be obtained from $G - \cup_{i=1}^{4} V(T_i)$ by adding a flat edge xr. Thus, every component of G' is SP-1. Let D' be an $i(G')$-set of G'. Therefore, $i(G') = |D'| \leq \frac{1}{3}(n - 12) = \frac{n}{3} - 4$. Note that $D' \cup \{x_2, x_3, x_4\}$ is an ID-set of G, so $i(G) \leq \frac{n}{3} - 1 < \frac{n}{3}$, showing a contradiction.

Case 3. $e(G[W]) = 3$. Let y_i' be the neighbor of y_i, and let z_i' be the neighbor of z_i, $i \in \{2, 3, 4\}$. Let $G' = G - \cup_{i=1}^{4} V(T_i)$. If G' is connected, then G' is an SP-1 graph, and $i(G') \leq \frac{n-12}{3} = \frac{n}{3} - 4$. If G' is disconnected, then it is assumed that G' consists of k components, and $G_1, G_2, \ldots, G_k$, so each component is SP-1. Thus, by Theorem 2.1, $i(G') = \sum_{i=1}^{k} i(G_i) \leq \sum_{i=1}^{k} \frac{1}{3}(n_2(G_i) + n_3(G_i)) = \frac{1}{3}(n_2(G') + n_3(G'))$. Let D' be an $i(G')$-set of G'. Then, D' can be extended to an ID-set of G by adding to it the vertices x_2, x_3, and x_4. It implies that $i(G) \leq |D'| + 3 \leq \frac{n}{3} - 4 + 3 = \frac{n}{3} - 1 < \frac{n}{3}$, showing a contradiction. □

Hence, every triangle unit belongs to a C_6^+-unit, i.e., the vertex $V(G)$ can be partitioned into sets each of which induces a C_6^+-unit in G. That is, $G \in \mathcal{H}$. □

4. Proof of Theorem 2.3

In this section, we give the proof of Theorem 2.3. Now we recall the content of Theorem 2.3: If G is an SP-2 graph without $(C_3\Box K_2)$-component, then $18i(G) \le 8n_2(G) + 5n_3(G)$.

Proof. Let $G = (V(G), E(G))$ be a counterexample SP-2 graph to Theorem 2.3 with a minimum order. Apparently, G is connected, and $18i(G) > 8n_2(G) + 5n_3(G)$. If $n = 4$, then G is K_4, which is not a counterexample. Similar to the argument of Theorem 2.1, there is no SP-2 graph when $n = 5$. If $n = 6$, G is one of the three graphs, as shown in Figure 4. In each of the cases, $18i(G) = 36 \le 8n_2(G) + 5n_3(G)$. Therefore, $n \ge 7$.

Then, the following useful fact is proved.

Fact 1. Let $X \subseteq V(G)$, $G' = G - X$, and let D' be an $i(G')$-set. If G' is an SP-2 graph, and there exists a set D such that $D' \cup D$ is an ID-set of G, then $18|D| > 8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G'))$.

Proof. By means of contradiction, it is assumed that $18|D| \le 8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G'))$. Since $D \cup D'$ is an ID-set of G and by the minimality of G, we have

$$18i(G) \le 18(|D| + |D'|) \le 18|D| + 8n_2(G') + 5n_3(G') \le 8n_2(G) + 5n_3(G).$$

It contradicts that G is a counterexample. □

The following claims are provided to describe some structural properties of G.

Claim 1. The following properties hold in G.

(1) The removal of flat edges of G cannot create an induced $K_{1,3}$, K_4^-, or C_6^+ subgraph;

(2) Adding flat edges on G to obtain a result graph with a maximum degree of three cannot create an induced $K_{1,3}$ or K_4^- subgraph.

Proof. (1) Let G' be obtained from G by removing some flat edges. Note that each vertex in $V(G')$ belongs to a triangle unit, and $\Delta(G) \le 3$, so G' has no induced $K_{1,3}$. To the contrary, suppose that G' has a K_4^--unit, say S, where $V(S) = \{a, b, c, d\}$ and ab is the missing edge in S. Since G has no K_4^+-unit, then $ab \in E(G)$, and ab is the removing flat edge. Thus, G is determined, and $G = K_4$, contradicting that $n \ge 7$. Similarly, if G' has a C_6^+-unit, then $G = C_3\Box P_2$, showing a contradiction.

(2) Let G' be obtained from G by adding some flat edges on G, $\Delta(G') = 3$. Since each vertex in $V(G')$ is still in a triangle unit and each edge in a K_4^--unit is a triangle edge, then G' is $\{K_{1,3}, K_4^-\}$-free. □

Claim 2. Any two triangle units have no common vertex.

Proof. Suppose T_1 and T_2 are two triangles with common vertices. If they have three common vertices, then $V(T_1) = V(T_2)$. If they have two common vertices, then $G = K_4$, or G has a K_4^--unit, showing a contradiction. If T_1 and T_2 have only one common vertex, say x, then $d(x) \ge 4$, contradicting that $\Delta(G) = 3$. □

Claim 3. No triangle in G has two vertices of degree two.

Proof. Suppose that there is a triangle unit T_1 with two vertices of degree two (see Figure 6a). Let $V(T_1) = \{x_1, y_1, z_1\}$, $d(y_1) = d(z_1) = 2$. Since $n \ge 7$, $d(x_1) = 3$, and x_1 is adjacent to a triangle T_2. Also, at most one of $V(T_2)$ has a degree of two since $n \ge 7$. Let $X = V(T_1)$, $G' = G - X$, so G' is connected and $\Delta(G) = 3$. Therefore, by Claim 1, G' is an SP-2 graph. Let D' be an $i(G')$-set of G', $D = \{y_1\}$, and then $D' \cup D$ is an ID-set of G. However, $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 18 = 18|D|$, contradicting to Fact 1. □

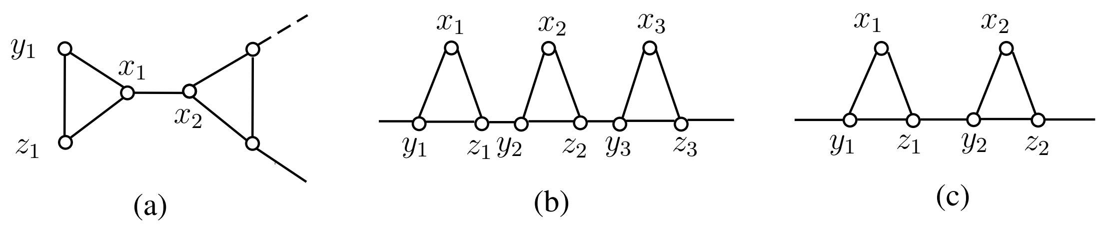


Figure 6. The graphs for Claims 3-5.

Claim 4. There are no 3 consecutive triangles with vertex of degree 2 in G.

Proof. Suppose T_1, T_2 and T_3 are three consecutive triangles with a vertex of degree two (see Figure 6b), where $V(T_i) = \{x_i, y_i, z_i\}$ and $d(x_i) = 2$, $i \in [3]$. If y_1 and z_3 are adjacent, then the graph G is determined. In this case, $18i(G) = 54 = 8n_2(G) + 5n_3(G)$, contradicting the fact that G is a counterexample to our theorem. Thus, $y_1z_3 \notin E(G)$. Let G' be the graph obtained from $G - V(T_2)$ by adding the edge z_1y_3. By Claim 1 and $\Delta(G') = 3$, G' is SP-2. Let D' be an $i(G')$-set of G'. If z_1 and y_3 are not contained in D', then $D' \cup \{x_2\}$ is an ID-set of G; otherwise, say $z_1 \in D'$, then $D' \cup \{z_2\}$ is an ID-set of G. This suggests that $18i(G) \leq 18(i(G') + 1) \leq 18 + 8(n_2(G) - 1) + 5(n_3(G) - 2) < 18i(G)$, showing a contradiction. □

Claim 5. There are no two consecutive triangles with a vertex of degree two in G.

Proof. Suppose T_1 and T_2 are two consecutive triangles with a vertex of degree two (see Figure 6c), where $V(T_i) = \{x_i, y_i, z_i\}$ and $d(x_i) = 2$, $i \in [2]$. Since $n \geq 7$, $y_1z_2 \notin E(G)$.

If y_1 and z_2 are adjacent to the same triangle (see Figure 7a), say T_3, where $V(T_3) = \{x_3, y_3, z_3\}$, $\{y_1y_3, z_2z_3\} \subseteq E(G)$, then $V(T_3)$ has no vertex of degree two by Claim 4, and x_3 is adjacent to another triangle T_4, where $V(T_4) = \{x_4, y_4, z_4\}$, $x_3x_4 \in E(G)$. Furthermore, T_4 has at most one vertex of degree two by Claim 3. In this case, let $X = \cup_{i=1}^{4} V(T_i)$, $G' = G - X$, and let D' be an $i(G')$-set of G'. Let $D = \{y_1, z_2, x_4\}$, and then $D \cup D'$ is an ID-set of G. Thus, if $V(T_4)$ has a vertex of degree two, then G' is a connected SP-2 graph, and $\Delta(G') = 3$, so $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 66 > 18|D|$, contradicting to Fact 1. Thus, each of $V(T_4)$ has a degree of three. If G' is not an SP-2 graph, then $G' = K_3$, and further there is a C_6^+-unit in G, a contradiction to the assumption that G is an SP-2 graph. Thus, G' is an SP-2 graph, and then $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 60 > 18|D|$. This contradicts Fact 1.

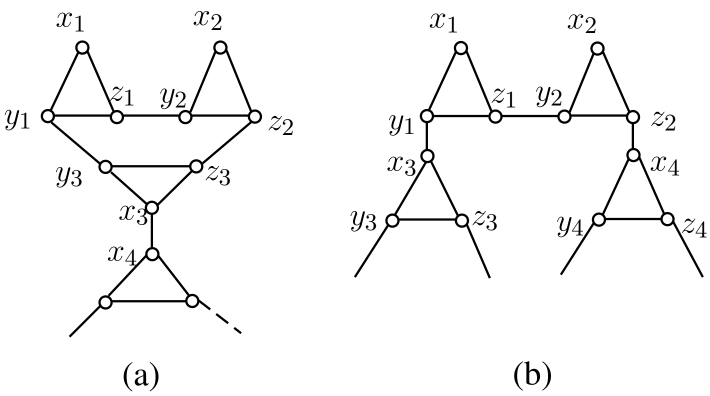


Figure 7. The graphs for Claim 5.

If y_1 is adjacent to a triangle T_3, z_2 is adjacent to a triangle T_4, $T_3 \neq T_4$ (see Figure 7b), where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{2, 3\}$, and $\{y_1x_3, z_2x_4\} \subseteq E(G)$. In this case, let G' be obtained from $G - V(T_1)$ by adding the edge x_3y_2. Therefore, G' is an SP-2 graph, and $G \neq C_3 \square P_2$. Let D' be an $i(G')$-set of G'. If x_3 and y_2 are not in D', then $D' \cup \{x_1\}$ is an ID-set of G; otherwise, say $x_3 \in D'$, then $D' \cup \{z_1\}$ is an ID-set of G. Thus, $18i(G) \leq 18(i(G') + 1) \leq 18 + 8(n_2(G) - 1) + 5(n_3(G) - 2) < 18i(G)$, showing a contradiction. □

Claim 6. G is cubic.

Proof. Assume a triangle T_1, where $V(T_1) = \{x_1, y_1, z_1\}$, contains a vertex x_1 of degree of two (see Figure 8a). Then, y_1 and z_1 have no common neighbor except for x_1 since G has no diamond unit and $n \geq 7$. Furthermore, they are not adjacent to the same triangle since G has no C_6^+-unit. Thus, it is assumed that y_1 and z_1 are adjacent to T_2 and T_3, respectively, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{2, 3\}$, $\{y_1x_2, z_1x_3\} \subseteq E(G)$. Each of $V(T_2 \cup T_3)$ has a degree of three by Claim 5.

If both y_2 and z_2 are not adjacent to y_3 or z_3, let G' be obtained from $G - V(T_1)$ by adding the edge x_2x_3. Thus, G' is an SP-2 graph, and $G \neq C_3 \square P_2$. Let D' be an $i(G')$-set of G'. If x_2 and x_3 are not in D', then $D' \cup \{x_1\}$ is an ID-set of G; otherwise, say $x_2 \in D'$, then $D' \cup \{z_1\}$ is an ID-set of G. Thus, $18i(G) \leq 18(i(G') + 1) \leq 18 + 8(n_2(G) - 1) + 5(n_3(G) - 2) < 18i(G)$, showing a contradiction.

Without loss of generality, it is assumed that $z_2y_3 \in E(G)$ (see Figure 8b), and then $y_2z_3 \notin E(G)$ since G has no C_6^+-unit. y_2 and z_3 are adjacent to the same triangle, say T_4, where $V(T_4) = \{x_4, y_4, z_4\}$, $\{y_2y_4, z_3z_4\} \subseteq E(G)$. If $d(x_4) = 2$, then graph G is determined. In this case, $8n_2(G) + 5n_3(G) = 66 > 54 = 18i(G)$, contradicting the fact that G is a counterexample to our theorem. Thus, $d(x_4) = 3$. Let $X = \cup_{i=1}^{4} V(T_i)$, $G' = G - X$, and then G' is an SP-2 graph by Claim 2 and Claim 3. Let $D = \{y_1, y_3, y_4\}$ and D' be an $i(G')$-set of G'. Hence, $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 60 > 54 = 18|D|$, showing a contradiction. So, it is assumed that y_2 is adjacent to T_4, and z_3 is adjacent to T_5 (see Figure 8c), where $V(T_4) = \{x_4, y_4, z_4\}$, $V(T_5) = \{x_5, y_5, z_5\}$, $y_2x_4, z_3x_5 \in E(G)$. Let $X = \cup_{i=1}^{4} V(T_i)$, $G' = G - X$, and let $D = \{y_1, y_3, x_4\}$, D' be an $i(G')$-set of G'. Thus, if $V(T_4)$ has a vertex of degree two, G' is an SP-2 graph without $C_3 \square K_2$ component, so $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 60 > 54 = 18|D|$, showing a contradiction. Thus, each of $V(T_4)$ has a degree of 3. If G' has no K_3 component, then G' is SP-2. In this case, $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 54 = 18|D|$, showing a contradiction. If G' has a K_3 component, i.e., y_4 and z_4 are adjacent to the same triangle T_6, then G contains a C_6^+-unit, showing a contradiction. □

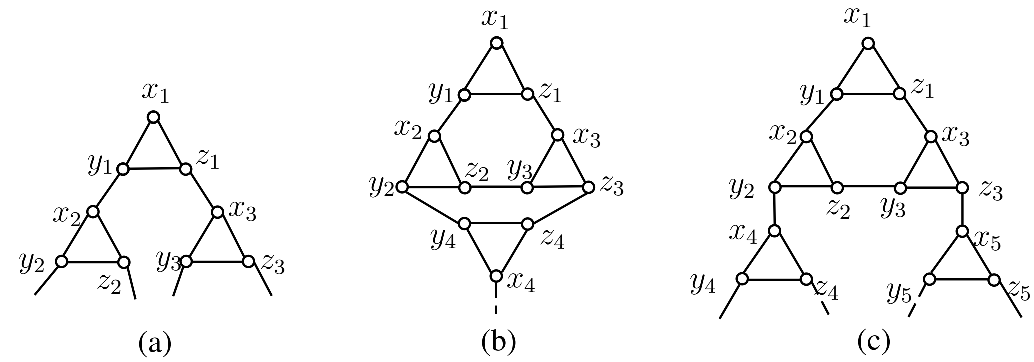


Figure 8. The graphs for Claim 6.

Claim 7. G contains no C_6 subgraph.

Proof. Suppose that G contains a C_6, $V(C_6) = \{v_1, v_2, \ldots, v_6\}$. The following two cases are considered.

Case 1. There exist triangles formed by the inner vertices in $V(C_6)$. Suppose that T_1 is a triangle, where $V(T_1) = \{v_1, v_2, v_3\}$. If the triangle where v_4 is located is also formed by three vertices inside $V(C_6)$, then $v_2v_4 \in E(G)$ or $v_4v_6 \in E(G)$. If $v_2v_4 \in E(G)$, then $G[\{v_1, v_2, v_3, v_4\}]$ is a K_4^--unit in G, showing a contradiction. If $v_4v_6 \in E(G)$, then $G[V(C_6)]$ is a C_6^+-unit, showing a contradiction. Thus, the triangle where v_4 is located consists of v_4, v_5, and a vertex outside $V(C_6)$ by Claim 2. Therefore, v_6 does not belong to a triangle, showing a contradiction.

Case 2. There is no triangle formed by the inner vertices inside $V(C_6)$. Hence, flat and triangle edges alternate along C_6 (see Figure 9a). Suppose T_1, T_2, and T_3 are three triangles containing vertex in $V(C_6)$, where $V(T_i) = \{x_i, y_i, z_i\}$ and x_i is a vertex outside C_6, $i \in [3]$. Given that G is C_6^+-free, then $x_ix_{i+1} \notin E(G)$, where $i + 1$ is understand modulo 3, $i \in [3]$. If x_i and x_{i+1} have a common neighbor, say x, then x is not in a triangle, showing a contradiction.

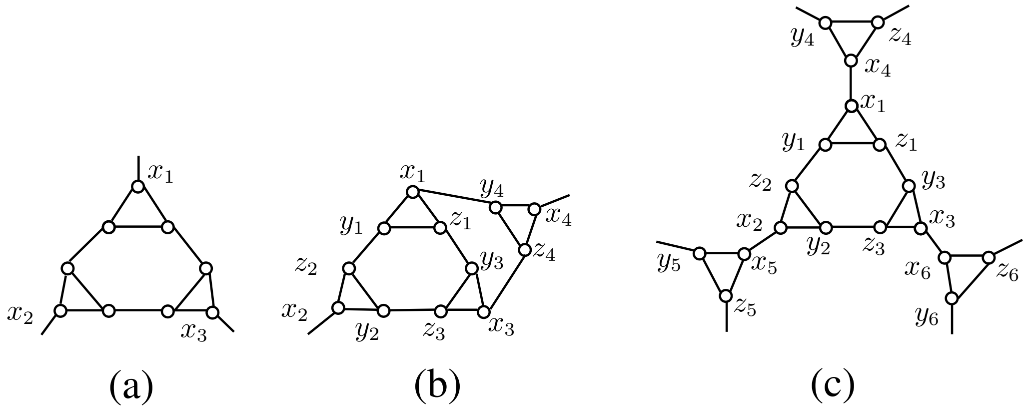


Figure 9. The graphs for Claim 7.

Next, it is claimed that x_i and x_{i+1} are not adjacent to the same triangle. By means of contradiction, suppose x_1 and x_3 are adjacent to the same triangle, say T_4 (see Figure 9b), where $V(T_4) = \{x_4, y_4, z_4\}$, and $x_1y_4, x_3z_4 \in E(G)$. If $x_2x_4 \in E(G)$, then G is determined, and $18i(G) \leq 54 < 5n_3(G) = 60$, contradicting that G is a counterexample graph. If $x_2x_4 \notin E(G)$, let $G' = G - \cup_{i=1}^{4} V(T_i)$, and D' is an $i(G')$-set of G'. By Claim 1, G' is an SP-2 graph. Let $D = \{z_2, y_3, y_4\}$, and then $D \cup D'$ is an ID-set of G. Thus, $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = -16 + 70 = 54 = 18|D|$, showing a contradiction.

So, x_1, x_2, and x_3 are adjacent to different triangles, respectively. Assume that x_i is adjacent to T_{i+3}, where $V(T_{i+3}) = \{x_{i+3}, y_{i+3}, z_{i+3}\}$ and $x_ix_{i+3} \in E(G)$, $i \in [3]$ (see Figure 9c). If both y_5 and z_5 are not adjacent to y_6 and z_6, let G' be obtained from $G - \cup_{i=1}^{4} V(T_i)$ by adding the flat edge x_5x_6. Note that G' is SP-2 and $G' \neq C_3 \square P_2$. Every $i(G')$-set of G' can be extended to be an ID-set of G by adding to it the vertices x_4, z_5 and y_6, Therefore, $18i(G) \leq 18(i(G') + 3) \leq 54 + 8(n_2(G) - 2) + 5(n_3(G) - 14) = 8n_2(G) + 5n_3(G) < 18i(G)$, showing a contradiction. Thus, it is assumed that $z_5y_6 \in E(G)$, and then $z_6y_5 \notin E(G)$; otherwise, $G[V(T_5) \cup V(T_6)]$ is a C_6^+-unit. By symmetry, $y_4y_5 \in E(G)$, and $z_4z_6 \in E(G)$. Therefore, G is determined. In this case, $18i(G) = 90 = 5n_3(G)$, contradicting the fact that G is a counterexample to our theorem. □

Claim 8. G contains no C_8 subgraph.

Proof. Suppose G contains a C_8, the following two cases are considered.

Case 1. There exist triangles formed by the inner vertices in $V(C_8)$. Since G has no K_4-unit, there are at most two disjoint triangles formed by $V(C_8)$.

Case 1.1. If there are two triangles formed by the inner vertices in $V(C_8)$, then there is a C_6 in $G[V(C_8)]$, showing a contradiction.

Case 1.2. There is a unique triangle T formed by the inner vertices in $V(C_8)$. For the renaming vertices except for T, we have $\{v_1, v_2, \ldots, v_5\} = V(C_8) \setminus V(T)$. For any $v_i \in V(C_8) \setminus V(T)$, the triangle where v_i is located cannot only contain v_i in $V(C_8)$; otherwise, $d(v_i) \geq 4$. By Claim 2, any two triangles have no common vertex, but it is impossible to have a partition in $V(C_8) \setminus V(T)$ such that each part has two vertices, showing a contradiction.

Case 2. There is no triangle formed by the inner vertices inside $V(C_8)$. That is, $V(C_8)$ is partitioned into 4 binary sets (see Figure 10a). In each binary set, these two vertices are adjacent, and otherwise, there is at least one 4^+-vertex, showing a contradiction. Any two vertices in $\{x_1, x_2, x_3, x_4\}$ are not adjacent since G is C_6-free.

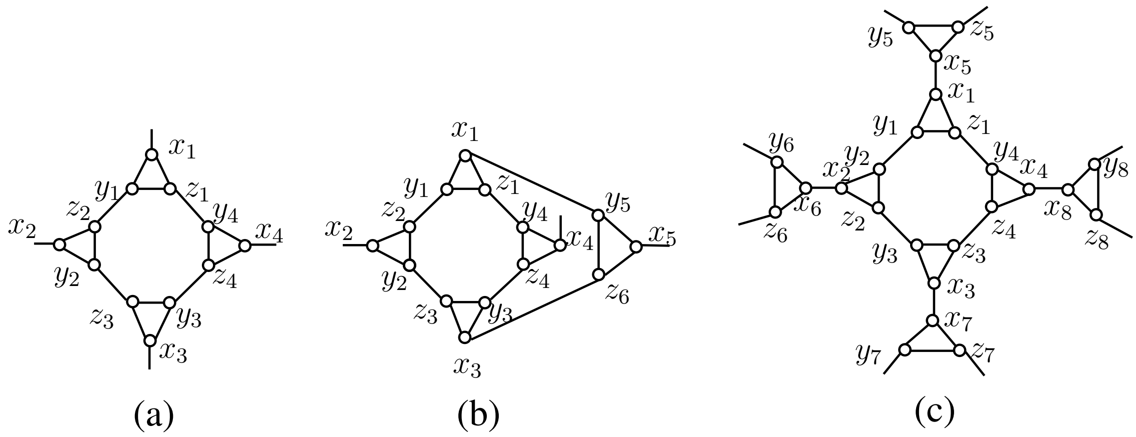


Figure 10. The graphs for Claim 8.

Claim 8.1. Any two vertices in $\{x_1, x_2, x_3, x_4\}$ are not adjacent to the same triangle.

Proof. First, x_i and x_{i+1} are not adjacent to the same triangle because G is C_6-free. By the symmetry, it is assumed that x_1 and x_3 are adjacent to the same triangle T_5 (see Figure 10b), where $V(T_5) = \{x_5, y_5, z_5\}$, $x_1y_5, x_3z_5 \in E(G)$. Note that $x_2x_5, x_4x_5 \notin E(G)$, since G has no C_6 subgraph.

Case A. x_5, x_2, x_4 are adjacent to the same triangle. Then, G is determined, and $i(G) = 5$, so $18i(G) = 90 = 5n$, showing a contradiction.

Case B. x_5 and x_2 are adjacent to the same triangle, and x_4 is adjacent to another triangle. Assume that x_2 and x_5 are adjacent to the same triangle T_6, where $V(T_6) = \{x_6, y_6, z_6\}$, $\{x_2y_6, x_5z_6\} \subseteq E(G)$. Meanwhile,

x_4 is adjacent to a triangle T_7, $T_7 \neq T_6$, where $V(T_7) = \{x_7, y_7, z_7\}$, $x_4x_7 \in E(G)$. Let G' be obtained from $G - V(T_1) \cup V(T_2) \cup V(T_4) \cup V(T_5)$ by adding the flat edges y_3y_6, x_3x_7, and let D' be an $i(G')$-set of G'. Since both T_6 and T_7 are not adjacent to T_3, G' has no C_6^+-unit. Hence, G' is an SP-2 graph, and $D' \cup \{z_2, y_4, y_5\}$ is an ID-set of G. Thus, $18i(G) \leq 18(i(G')+3) \leq 54+8(n_2(G)+2)+5(n_3(G)-14) = 8n_2(G)+5n_3(G) < 18i(G)$, showing a contradiction.

Case C. x_5 is not adjacent to the same triangle with x_2 or x_4. Assume that x_2 is adjacent to a triangle T_6, and x_5 is adjacent to a triangle T_7, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{6, 7\}$. Let G' be obtained from $G - V(T_1) \cup V(T_2) \cup V(T_4) \cup V(T_5)$, and let D' be an $i(G)$-set of G'. Since both T_6 and T_7 are not adjacent to T_3, G' has no C_6^+-unit. Hence, G' is an SP-2 graph, and $D' \cup \{z_2, y_4, x_5\}$ is an ID-set of G. Therefore, $18i(G) \leq 18(i(G') + 3) \leq 54 + 8(n_2(G) + 2) + 5(n_3(G) - 14) = 8n_2(G) + 5n_3(G) < 18i(G)$, showing a contradiction. □

Thus, it is assumed that x_i is adjacent to triangle T_{i+4}, where $V(T_{i+4}) = \{x_{i+4}, y_{i+4}, z_{i+4}\}$, $i \in [4]$, $\{x_1x_5, x_2x_6, x_3x_7, x_4x_8\} \subseteq E(G)$ (see Figure 10c). Let G' be obtained from $G - V(T_1) \cup V(T_2) \cup V(T_4) \cup V(T_5)$ by adding the flat edges y_3x_6, z_3x_8, and D' is an $i(G')$-set of G'. Since both T_6 and T_8 are not adjacent to T_3, G' has no C_6^+-unit. Thus, G' is an SP-2 graph.

Hence, $18i(G) \leq 18(i(G') + 3) \leq 54 + 8(n_2(G) + 2) + 5(n_3(G) - 14) = 8n_2(G) + 5n_3(G) < 18i(G)$, showing a contradiction. □

Therefore, G is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph, and G has no C_6 or C_8 as subgraph. Fix a triangle $T_1 \subseteq V(G)$, where $V(T_1) = \{x_1, y_1, z_1\}$ (see Figure 11). x_1, y_1, and z_1 have no common neighbors because G is K_4^--free. Any two vertices in $V(T_1)$ can not be adjacent to the same triangle since G is C_6^+-free and $G \neq C_3 \Box P_2$. Hence, it is assumed that x_1, y_1, and z_1 are adjacent to T_2, T_3, and T_4, respectively, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{2, 3, 4\}$, and $\{x_1x_2, y_1x_3, z_1x_4\} \subseteq E(G)$. For any $i, j \in \{2, 3, 4\}$, $i \neq j$, both y_i and z_i are not adjacent to y_j or z_j because G has no C_6 subgraph. y_i and z_j can not be adjacent to the same triangle because G is C_8-free. Thus, y_2, z_2, y_3, z_3, y_4, and z_4 are adjacent to different triangles, respectively. Suppose that y_2, z_2, y_3, z_3, y_4, and z_4 are adjacent to T_5 T_6, T_7, T_8, T_9, and T_{10}, respectively, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{5, 6, 7, 8, 9, 10\}$, $\{y_2x_5, z_2x_6, y_3x_7, z_3x_8, y_4x_9, z_4x_{10}\} \subseteq E(G)$. Let G' be obtained from $G - \cup_{i=1}^4 V(T_i)$, and D' be an $i(G')$-set. By Claim 1 and Claim 7, G' is SP-2. Noting that $D' \cup \{x_2, x_3, x_4\}$ is an ID-set of G, so $18i(G) \leq 18(i(G') + 3) \leq 54 + 5(n_3(G) - 12) = 8n_2(G) + 5n_3(G) - 6 < 18i(G)$, showing a contradiction.

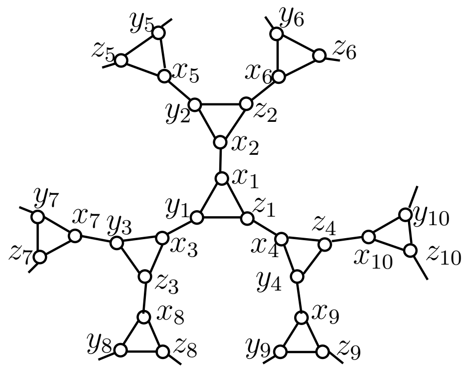


Figure 11. Illustrations for G.

This completes the proof. □

Corollary 2.4 *If G is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph with no $(C_3 \Box K_2)$-component, then $i(G) \leq \frac{5}{18}|V(G)|$.*

Proof. If G is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph with no $(C_3 \Box K_2)$-component, then G is an SP-2 graph, and each vertex in $V(G)$ is of degree three. Thus, by Theorem 2.3, $i(G) \leq \frac{5}{18}|V(G)|$. □

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